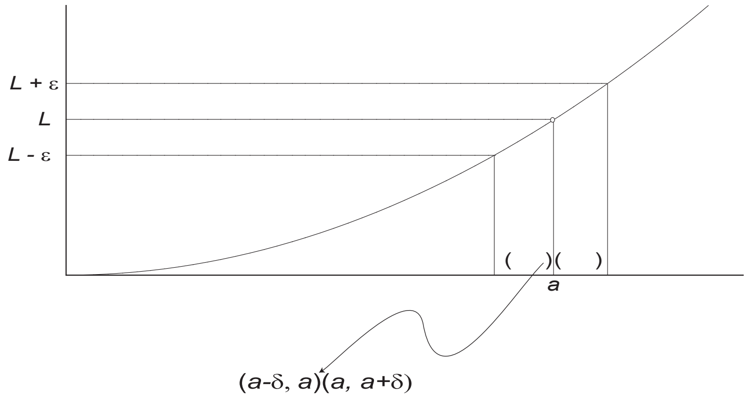
Appendix 1.1

1. **Definition of limit**. Recall from course MATH10242 that we used the definition of convergence of a sequence to test a given sequence converges by assuming that an $\varepsilon > 0$ is given and then trying to find an appropriate N. Similarly, we will check a given function has limit L at a point a by **assuming** that some $\varepsilon > 0$ is given and then **trying to find** an appropriate $\delta > 0$.



2. **Deleted neighbourhood.** Proof of

$$(a-\delta, a) \cup (a, a+\delta) = \{x : 0 < |x-a| < \delta\}.$$

Proof i) To prove $(a-\delta, a) \cup (a, a+\delta) \subseteq \{x : 0 < |x-a| < \delta\}$.

Let $x \in (a-\delta, a) \cup (a, a+\delta)$. This implies

$$\begin{aligned} &\implies x \in (a-\delta, a) \quad \text{or} \quad x \in (a, a+\delta), \\ &\implies a-\delta < x < a \quad \text{or} \quad a < x < a+\delta \\ &\implies 0 < |x-a| < \delta \quad \text{or} \quad 0 < |x-a| < \delta. \end{aligned}$$

i.e. in both cases $0 < |x-a| < \delta$. Hence $(a-\delta, a) \cup (a, a+\delta) \subseteq \{x : 0 < |x-a| < \delta\}$.

ii) To prove $\{x : 0 < |x-a| < \delta\} \subseteq (a-\delta, a) \cup (a, a+\delta)$.

Assume $x \in \{x : 0 < |x-a| < \delta\}$ so $0 < |x-a| < \delta$. We have two cases.

a) Assume $x > a$. Then $x - a > 0$ in which case $|x-a| = x-a$ and thus $0 < x-a < \delta$. Reinterpret this as $x \in (a, a+\delta)$, which can be weakened to $x \in (a-\delta, a) \cup (a, a+\delta)$.

b) Assume $x < a$.Then $x - a < 0$ in which case $|x - a| = a - x$ and thus $0 < a - x < \delta$. Reinterpret this as $x \in (a - \delta, a)$, which can be weakened to $x \in (a - \delta, a) \cup (a, a + \delta)$.

In both cases $x \in (a - \delta, a) \cup (a, a + \delta)$. Hence $\{x : 0 < |x - a| < \delta\} \subseteq (a - \delta, a) \cup (a, a + \delta)$.

Combine the two set inclusions as

$$(a - \delta, a) \cup (a, a + \delta) = \{x : 0 < |x - a| < \delta\}.$$

■

3. **The triangle inequality**.

Lemma *For $a, b \in \mathbb{R}$,*

$$|a - b| \geq ||a| - |b||.$$

Proof Given $a, b \in \mathbb{R}$ use the triangle inequality within

$$|a| = |a - b + b| \leq |a - b| + |b|,$$

which rearranges to give

$$|a - b| \geq |a| - |b|.$$

Alternatively, starting with b and not a,

$$|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a|,$$

which rearranges to give

$$|a - b| \geq |b| - |a|.$$

We can combine these two lower bounds as

$$|a - b| \geq ||a| - |b||,$$

where the right hand side is now always positive due to the modulus sign. ■

4. **Limits of polynomials** We will see soon in the lectures that for a polynomial $p(x)$ we have $\lim_{x\to a} p(x) = p(a)$. So when we are trying to verify the ε - δ definition of limit here we need show that $|p(x) - p(a)|$ is small for x close to a. But when $x = a$ then $p(a) - p(a) = 0$, i.e. $x = a$ is a root of $p(x) - p(a)$. In turn this means that $x - a$ is a factor of $p(x) - p(a)$, i.e. $p(x) - p(a) = (x - a)\, q(x)$ for some polynomial $q(x)$.

   This was seen above in the example

   $$\lim_{x\to 2} \left(x^3 + x^2 - 4x\right) = 4.$$

   Here $p(x) = x^3 + x^2 - 4x$ and $p(2) = 4$. So

   $$p(x) - p(2) = x^3 + x^2 - 4x - 4 = (x - 2)\left(ax^2 + bx + c\right).$$

   Equating coefficients, $a = 1$, $c = 2$ and $b = 3$, and so $q(x) = x^2 + 3x + 2$.

5. **Example** 1.1.10 *By verifying the* ε - δ *definition show that*

   $$\lim_{x\to 2} \frac{x^2 + 2x + 2}{x + 3} = 2.$$

   Solution To verify this we need consider

   $$\frac{x^2 + 2x + 2}{x + 3} - 2 = \frac{x^2 - 4}{x + 3}.$$

   The numerator will always have a factor of $x - a$, here $x - 2$. (Why?) In this case

   $$\frac{x^2 + 2x + 2}{x + 3} - 2 = (x - 2)\,\frac{x + 2}{x + 3}.$$

   We are bounding $x - 2$ by $|x - 2| < \delta$. For our example assume $\delta \le 1$ so $|x - 2| < \delta \le 1$ becomes $-1 < x - 2 < 1$, i.e. $1 < x < 3$. We have to check that $(x + 2) / (x + 3)$ is not too large in this interval.

   Method 1. With linear polynomials on top and bottom it is easy to write

   $$\frac{x + 2}{x + 3} = \frac{x + 3 - 1}{x + 3} = 1 - \frac{1}{x + 3}.$$

   Then

   $$\begin{aligned} 1 < x < 3 \quad &\Longrightarrow \quad 4 < x + 3 < 6 \\ &\Longrightarrow \quad \frac{1}{6} < \frac{1}{x + 3} < \frac{1}{4} \\ &\Longrightarrow \quad \frac{3}{4} = 1 - \frac{1}{4} < 1 - \frac{1}{x + 3} < 1 - \frac{1}{6} = \frac{5}{6} \end{aligned}$$

Hence

$$\left|\frac{x+2}{x+3}\right| < \frac{5}{6},$$

and we can choose $\delta = \min(1, 6\varepsilon/5)$ when verifying the ε - δ definition of

$$\lim_{x\to 2} \frac{x^2+2x+2}{x+3} = 2.$$

Method 2. This method is overkill for such a simple rational function. It is based on the observations that

$$\max_{[a,b]} \frac{f(x)}{g(x)} \le \frac{\max_{[a,b]} f(x)}{\min_{[a,b]} g(x)} \quad \text{and} \quad \min_{[a,b} \frac{f(x)}{g(x)} \ge \frac{\min_{[a,b]} f(x)}{\max_{[a,b]} g(x)},$$

as long as $f(x) \ge 0$ and $g(x) > 0$ on $[a,b]$.

For our example assume $\delta \le 1$ so $|x-2| < \delta \le 1$ becomes $-1 < x-2 < 1$, i.e. $1 < x < 3$. Then

$$\frac{x+2}{x+3} \le \frac{3+2}{1+3} = \frac{5}{4} \quad \text{and} \quad \frac{x+2}{x+3} \ge \frac{1+2}{3+3} = \frac{1}{2}.$$

This suggests we can choose $\delta = \min(1, 4\varepsilon/5)$ when verifying the ε - δ definition of

$$\lim_{x\to 2} \frac{x^2+2x+2}{x+3} = 2.$$

Check this is so.

6. **Example 1.1.14** *Prove by contradiction that*

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$$

does not exist.

Solution Assume for a contradiction that $\lim_{x\to 0} \sin(\pi/x)$ exists. Let $L = \lim_{x\to 0} \sin(\pi/x)$.

Choose $\varepsilon = 1/2$ in the ε - δ definition of limit to find $\delta > 0$ such that if $0 < |x| < \delta$ then

$$\left|\sin\left(\frac{\pi}{x}\right) - L\right| < \frac{1}{2}. \tag{7}$$

Choose $n \in \mathbb{N}$ so large that $x_1 = 2/(1+4n) < \delta$. For such x_1 we have that (7) holds while $\sin(\pi/x_1) = 1$ and so

$$|1 - L| < \frac{1}{2}. \tag{8}$$

Next choose $n \in \mathbb{N}$ so large that $x_2 = 2/(3+4n) < \delta$. For such x_2 we have that (7) holds while $\sin(\pi/x_2) = -1$ and so

$$|-1 - L| < \frac{1}{2}, \quad \text{i.e.} \quad |1 + L| < \frac{1}{2}. \tag{9}$$

We combine (9) and (8) using the triangle inequality as

$$2 = |1 - L + 1 + L| \leq |1 - L| + |1 + L| < \frac{1}{2} + \frac{1}{2} = 1.$$

Contradiction, hence assumption false, thus $\lim_{x\to 0} \sin(\pi/x)$ does **not** exist. ■

I leave it to the student to write out this proof, changing $\sin(\pi/x)$ to $x/|x|$ to show that

$$\lim_{x\to 0} \frac{x}{|x|} \quad \text{does not exist.}$$

7. **Uniqueness of limits.** In lectures it was shown that if $\lim_{x\to a} f(x)$ exists then it is unique. The $x \to a$ can be replaced by any of $x \to a+$, $x \to a-$, $x \to +\infty$ or $x \to -\infty$.

   Example *If* $\lim_{x\to+\infty} f(x)$ *exists, then the limit is unique.*

   Solution Assume that for the function f the limit is **not** unique. Let $\ell_1 < \ell_2$ be two of the different limit values (there may be more than two). In the ε - X definition of $\lim_{x\to+\infty} f(x)$ choose

$$\varepsilon = \frac{\ell_2 - \ell_1}{3} > 0.$$

   Then from definition of $\lim_{x\to+\infty} f(x) = \ell_1$ we find $X_1 > 0$ such that $x > X_1$ implies

$$|f(x) - \ell_1| < \varepsilon. \tag{10}$$

   Similarly, from the definition of $\lim_{x\to+\infty} f(x) = \ell_2$ we find $X_2 > 0$ such that $x > X_2$ implies

$$|f(x) - \ell_2| < \varepsilon. \tag{11}$$

Choose an $x_0 > \max(X_1, X_2)$. For such a point both (10) and (11) hold. Hence

$$\begin{aligned} |\ell_2 - \ell_1| &= |\ell_2 - f(x_0) + f(x_0) - \ell_1| \\ &\leq |\ell_2 - f(x_0)| + |f(x_0) - \ell_1| \\ &\qquad \text{by the triangle inequality,} \\ &< \varepsilon + \varepsilon \quad \text{by (10) and (11)}, \\ &= 2\varepsilon \\ &= 2|\ell_2 - \ell_1|/3. \end{aligned}$$

Dividing through by $|\ell_2 - \ell_1| \neq 0$ we get $1 < 2/3$, a contradiction. Hence the assumption is false and so, if it exists, $\lim_{x\to+\infty} f(x)$ is unique. ■

I leave it to the student to check that, if it exists, then $\lim_{x\to-\infty} f(x)$ is unique.

8. I stated in the lectures that the $\varepsilon - \delta$ definition of limit at a and the alternative definition were of limit in terms of sequences were equivalent. I prove this now.

Theorem 1.1.11 *For $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$,*

$$\lim_{x\to a} f(x) = L$$

iff for all sequences $\{x_n\}_{n\geq1}$ for which $x_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = a$ we have

$$\lim_{n\to\infty} f(x_n) = L.$$

Proof ($\Leftarrow$) We are assuming there exists $L \in \mathbb{R}$ such that for all sequences $\{x_n\}_{n\geq1}$ for which $x_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = a$ we have $\lim_{n\to\infty} f(x_n) = L$. The proof continues by contradiction. Assume that $\lim_{x\to a} f(x) = L$ is false for this value of L. Symbolically this means

$$\exists\varepsilon > 0, \forall\delta > 0, \exists x \in A : 0 < |x - a| < \delta \text{ and } |f(x) - L| \geq \varepsilon. \quad (12)$$

Because this assures us of the existence of a *particular* ε, call it ε_0.

Apply (12) repeatedly with $\delta = 1/n$ to find for each $n \geq 1$ a point $x_n \in A$ with $0 < |x_n - a| < 1/n$ and $|f(x_n) - L| \geq \varepsilon_0$. Because of $|x_n - a| < 1/n$ we have that $\lim_{n\to\infty} x_n = L$. Because of $0 < |x_n - a|$ we have that $x_n \neq a$ for all $n \geq 1$. Hence, by our initial assumption we have $\lim_{n\to\infty} f(x_n) = L$. From the definition of convergence for a sequence with $\varepsilon = \varepsilon_0/2$, this means there exists $N \geq 1$ such $|f(x_n) - L| < \varepsilon_0/2$ for all $n \geq N$. Yet a deduction from (12) was that $|f(x_n) - L| \geq \varepsilon_0$ for all $n \geq 1$. This contradiction means that our last assumption is false, and so $\lim_{x\to a} f(x) = L$ holds.

($\Rightarrow$) Assume $\lim_{x\to a} f(x) = L$. Let $\{x_n\}_{n\geq 1}$ be a sequence for which $x_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = a$. Let $\varepsilon > 0$ be given. From the definition of $\lim_{x\to a} f(x) = L$ we get that there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon. \tag{13}$$

Choose $\varepsilon = \delta$ in the definition of $\lim_{n\to\infty} x_n = a$ to find $N \geq 1$ such that if $n \geq N$ then $|x_n - a| < \delta$. Since we are assuming $x_n \neq a$ for all $n \geq 1$ this gives $0 < |x_n - a| < \delta$. Then, by (13), $|f(x_n) - L| < \varepsilon$. So we have shown that

$$\forall \varepsilon > 0, \exists N \geq 1, n \geq N \Longrightarrow |f(x_n) - L| < \varepsilon.$$

That is, we have verified the definition of $\lim_{n\to\infty} f(x_n) = L$. ■